
Algebraic multilevel preconditioner with projectors

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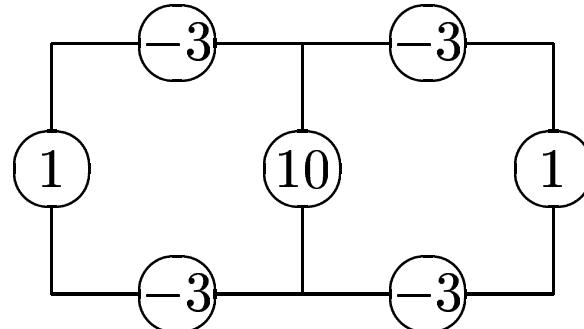
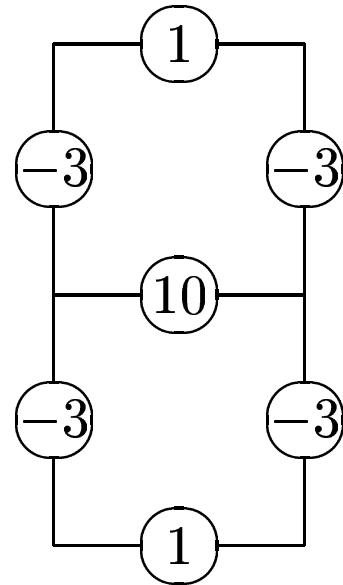
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Motivation

We want to attack the following problems:

- problems resulting in stiff SPD systems (in particular, SPD systems for edge-based intensities in mixed FE methods);



Motivation

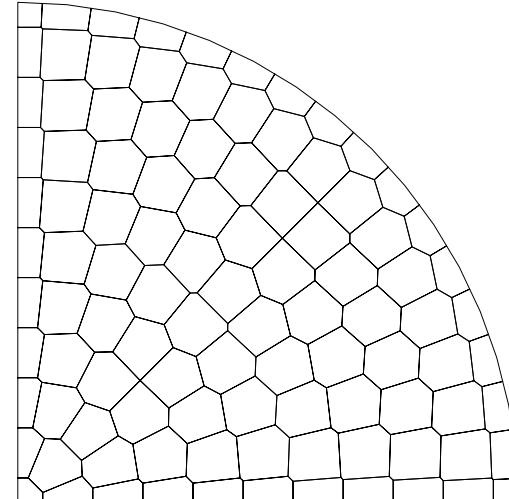
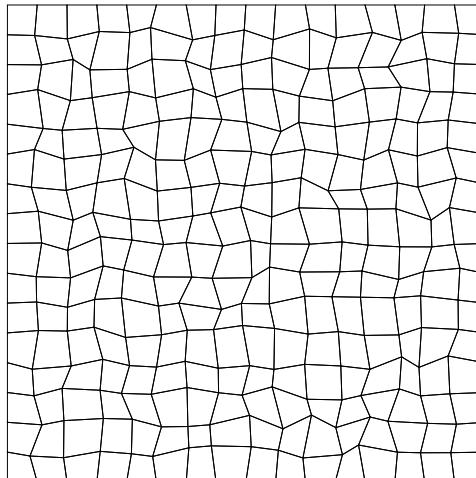
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- problems on structured and unstructured meshes having elements of mixed types (e.g., Voronoï meshes).



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We want to attack the following problems:

- problems resulting in stiff SPD systems (in particular, SPD systems for edge-based intensities in mixed FE methods);
- problems with materials having fine heterogeneous structure (e.g., different diffusion coefficients in every mesh cell);
- problems on structured and unstructured meshes having elements of mixed types (e.g., Voronoï meshes).

For simplicity, we consider the elliptic equation

$$-\operatorname{div}(K \nabla u) = f \quad \text{in } \Omega$$

with Neumann b.c. on $\partial\Omega$.

Two-level preconditioner (1/5)

Matrix \mathcal{A} can be written in an assembling form

$$\mathcal{A} = \sum_{i=1}^m \mathcal{N}_i A_i \mathcal{N}_i^T$$

where A_i is an elemental matrix and \mathcal{N}_i is an assembling matrix.

The preconditioner \mathcal{B} is given by

$$\mathcal{B} = \sum_{i=1}^m \mathcal{N}_i B_i \mathcal{N}_i^T$$

where

$$B_i \sim A_i.$$

Two-level preconditioner (2/5)

Consider an eigenvalue problem

$$A_i w = \lambda M_i w, \quad M_i = M_i^T > 0.$$

Then,

$$A_i = M_i W_i \Lambda_i W_i^T M_i$$

where

$$\Lambda_i = \text{diag}\{\lambda_1^i, \dots, \lambda_s^i\}, \quad \lambda_1^i = 0,$$

and W_i is a matrix of eigenvectors. Define

$$B_i = M_i W_i \tilde{\Lambda}_i W_i^T M_i$$

where

$$\tilde{\Lambda}_i = \text{diag}\{0, 1, \dots, 1\}.$$

Two-level preconditioner (3/5)

- Let $M_i = \text{diag}(A_i)$;
- Then, B_i is spectrally equivalent to A_i ,

$$\lambda_2^i(B_i x, x) \leq (A_i x, x) \leq \lambda_s^i(B_i x, x) \quad \forall x \in \Re^s,$$

and λ_s^i/λ_2^i depends only on the element shape and tensor anisotropy;

- Define $P_i = w_1^i [w_1^i]^T$,

$$P_i = \alpha_i \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Then,

$$B_i = M_i - M_i P_i M_i.$$

Two-level preconditioner (4/5)

Theorem.

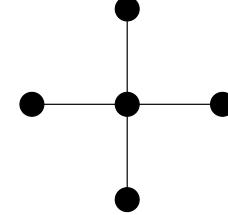
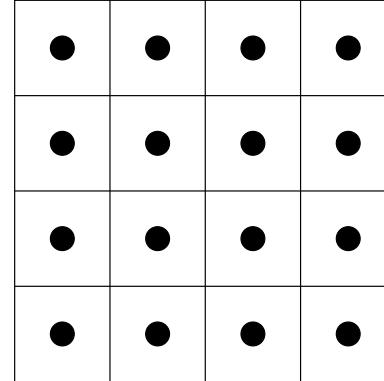
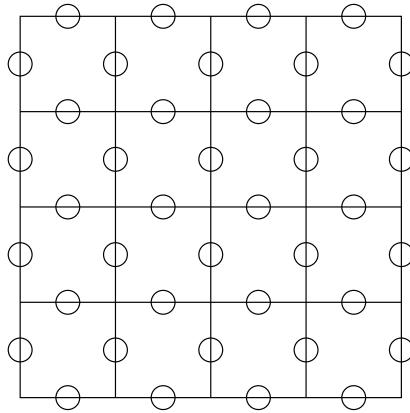
$$c_1(\mathbf{B}x, x) \leq (\mathbf{A}x, x) \leq c_2(\mathbf{B}x, x) \quad \forall x \in \mathbb{R}^n$$

where c_1 and c_2 depend only on the shape of mesh cells and tensor anisotropy.

$$\begin{aligned}\mathbf{B} = \sum_{i=1}^m \mathcal{N}_i B_i \mathcal{N}_i^T &= \sum_{i=1}^m \mathcal{N}_i (M_i - M_i P_i M_i) \mathcal{N}_i^T \\ &= \mathbf{M} - \sum_{i=1}^m \mathcal{N}_i M_i P_i M_i \mathcal{N}_i^T \equiv \mathbf{M} - \mathbf{C}.\end{aligned}$$

- rank $\mathbf{C} = m < n = \text{rank } \mathbf{B}$;
- \mathbf{B} is a M -matrix;
- $P_i M_i$ is M_i -orthogonal projector onto kernels of both A_i and B_i .

Two-level method (5/5)



Consider the system $\mathbf{B}v = (\mathbf{M} - \mathbf{C})v = g$. Let

$$\hat{v}_i = [w_1^i]^T M_i \mathcal{N}_i^T v \quad \text{and} \quad \hat{g}_i = [w_1^i]^T M_i \mathcal{N}_i^T \mathbf{M}^{-1} g.$$

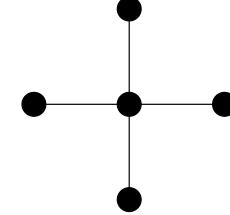
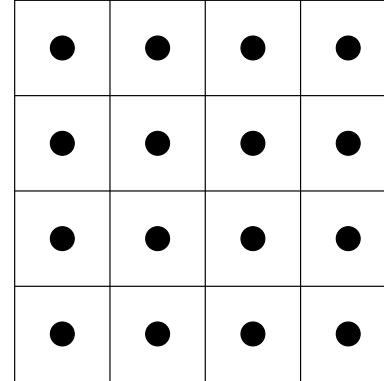
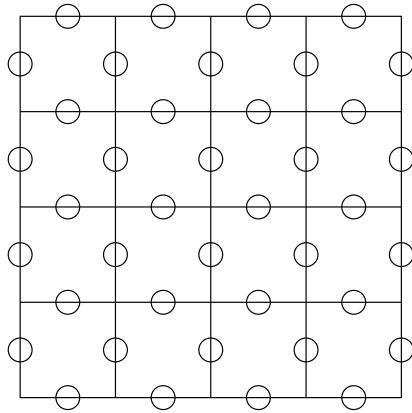
Then, the above system results in

$$(I - \hat{Q})\hat{v} = \hat{g}, \quad \hat{Q} \in \Re^{m \times m},$$

where $\hat{Q} = (\hat{q}_{ij})$ and

$$\hat{q}_{ij} = [w_1^i]^T M_i \mathcal{N}_i^T \mathbf{M}^{-1} \mathcal{N}_j M_j w_1^j.$$

Two-level method (5/5)



Consider the system $\mathbf{B}v = (\mathbf{M} - \mathbf{C})v = g$. Let

$$\hat{v}_i = [w_1^i]^T M_i \mathcal{N}_i^T v \quad \text{and} \quad \hat{g}_i = [w_1^i]^T M_i \mathcal{N}_i^T \mathbf{M}^{-1} g.$$

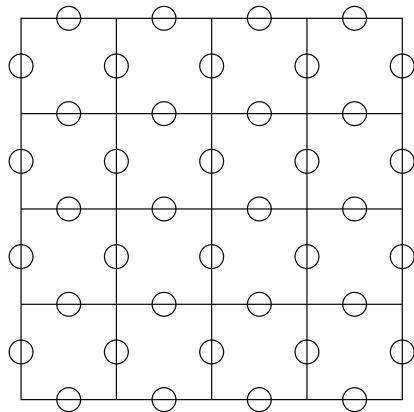
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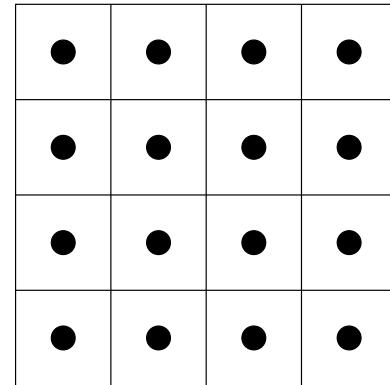
where $\hat{Q} = (\hat{q}_{ij})$ and

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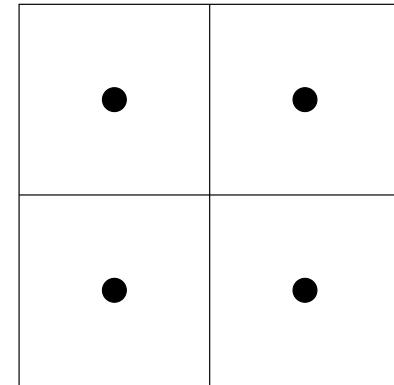
Multi-level method (1/3)



A



$B = A_1 = M - C_1$



$A_2 = M - C_2$

Recall, that

$$A_1 = \sum_{i=1}^m \mathcal{N}_{1,i} (M_{1,i} - M_{1,i} P_{1,i} M_{1,i}) \mathcal{N}_{1,i}^T.$$

Define

$$A_2 = \sum_{i=1}^{m/4} \mathcal{N}_{2,i} (M_{2,i} - M_{2,i} P_{2,i} M_{2,i}) \mathcal{N}_{2,i}^T$$

where

$$M_{2,i} = \sum_j \hat{\mathcal{N}}_{1,j} M_{1,j} \hat{\mathcal{N}}_{1,j}^T.$$

Multi-level method (1/3)

The figure displays three matrices labeled A , $B = A_1 = M - C_1$, and $A_2 = M - C_2$. Matrix A is a 16x16 grid of open circles. Matrix B is a 4x4 grid of solid black dots, with the bottom-left 2x2 subgrid shaded with diagonal lines. Matrix A_2 is a 2x2 grid of solid black dots, with the bottom-left subgrid shaded with diagonal lines.

$$A$$
$$B = A_1 = M - C_1$$
$$A_2 = M - C_2$$

Recall, that

$$A_1 = \sum_{i=1}^m \mathcal{N}_{1,i} (M_{1,i} - M_{1,i} P_{1,i} M_{1,i}) \mathcal{N}_{1,i}^T.$$

Define

$$A_2 = \sum_{i=1}^{m/4} \mathcal{N}_{2,i} (M_{2,i} - M_{2,i} P_{2,i} M_{2,i}) \mathcal{N}_{2,i}^T$$

where

$$M_{2,i} = \sum_j \hat{\mathcal{N}}_{1,j} M_{1,j} \hat{\mathcal{N}}_{1,j}^T.$$

Multi-level method (2/3)

Repeating the coarsening algorithm, we get a sequence of full rank matrices $\textcolor{blue}{A}, \textcolor{blue}{A}_1, \textcolor{blue}{A}_2, \dots, \textcolor{blue}{A}_L$ such that

$$\textcolor{blue}{A}_l = \textcolor{blue}{M} - \textcolor{blue}{C}_l, \quad l = 1, \dots, L.$$

Define space $V_l = \text{im} \textcolor{blue}{C}_l$. Then

- $\dim V_l = \text{rank } \textcolor{blue}{C}_l = m_l \approx m_{l-1}/4$;
- $V_L \subset V_{L-1} \subset \dots \subset V_1 \subset \mathbb{R}^n$;

Lemma. Let $\mathcal{T}_{l,h}$ be a sequence of square nested meshes. Then

$$\frac{c_1}{2^l} (A_l x, x) \leq (A_{l-1} x, x) \leq c_2 (A_l x, x) \quad \forall x \in V_{l-1}$$

where c_1 and c_2 are independent of l and the discretization parameter h .

Multi-level method (3/3)

In order to evaluate $\tilde{r} = \textcolor{red}{H}_l^{-1}r$, $r \in V_{l-1}$, we proceed as follows:

- take a special initial guess:

$$x^0 = \textcolor{blue}{M}^{-1}r;$$

- enter a subspace:

$$\xi^0 = \textcolor{blue}{A}_l x^0 - r = -\textcolor{blue}{C}_l \textcolor{blue}{M}^{-1}r, \quad y^0 = -\xi^0, \quad \xi^0 \in V_l;$$

- iterate in the subspace :

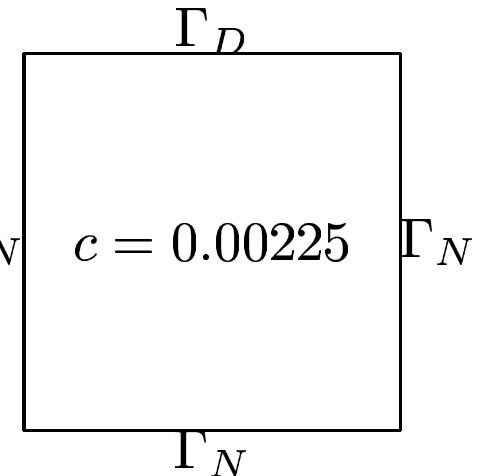
$$\begin{aligned}\xi^i &= \xi^{i-1} - \gamma_i \textcolor{blue}{A}_l \textcolor{red}{H}_{l+1}^{-1} \xi^{i-1} \\ y^i &= y^{i-1} - \gamma_i \textcolor{blue}{C}_l \textcolor{red}{H}_{l+1}^{-1} \xi^{i-1}, \quad i = 1, \dots, \beta;\end{aligned}$$

- leave the subspace:

$$\tilde{r} = \textcolor{blue}{M}^{-1}(y^\beta + r).$$

Numerical experiments (1/2)

$$\begin{aligned}-\operatorname{div}(K \nabla p) + c p &= 1 && \text{in } \Omega \\ p &= 0 && \text{on } \Gamma_D \\ (K \nabla p) \mathbf{n} &= 0 && \text{on } \Gamma_N\end{aligned}$$

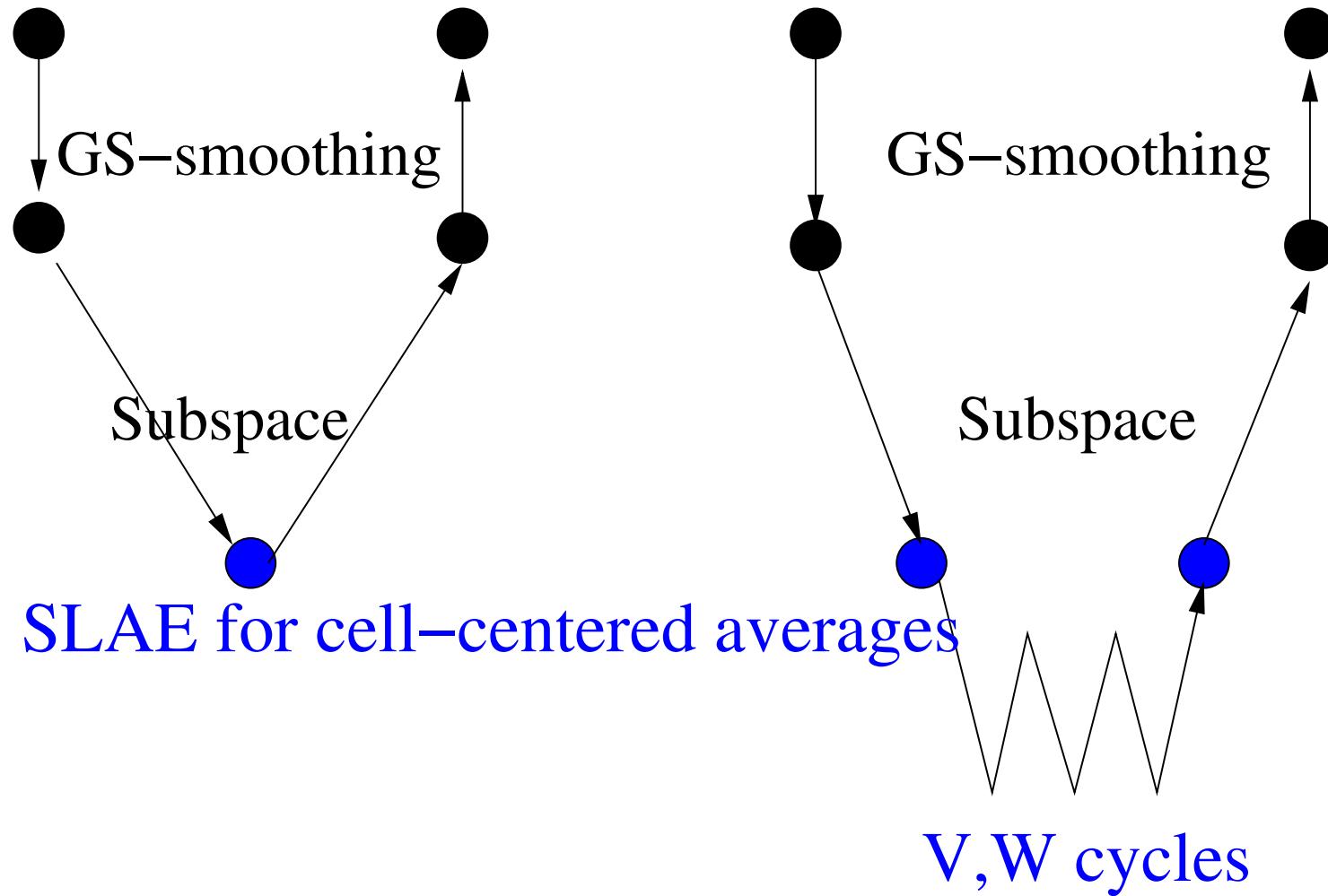


Components used in numerical experiments:

- Jacobi preconditioner;
- W-cycle of Multilevel preconditioner with projectors;
- W-cycle of Black Box MG (J.Dendy, D.Moulton, release 1.5.2, 2001);
- V-cycle of AMG (J.Ruge, K.Stüben, R.Hempel, release 1.5, 1990).

Numerical experiments (2/2)

SLAE for edge-based intensities



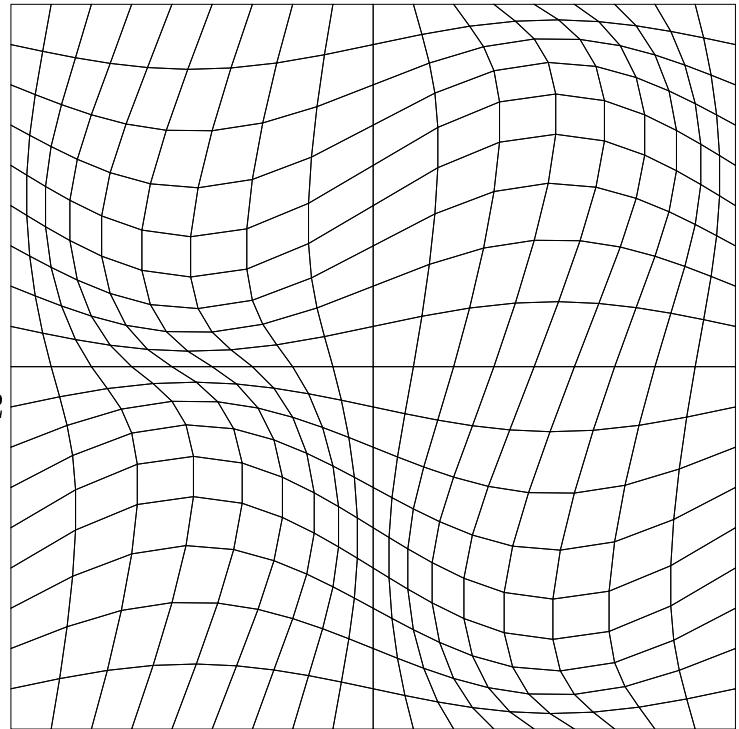
Smooth grids (1/3)

- grids are generated via a smooth mapping of reference square grids;
- we consider both mimetic FD and mixed FE discretizations.

1. $K = K_1 = 1$;
2. $K = K_2(x)$ where

$$K_2 = \begin{cases} 10^3, & x \in (0; 0.5)^2 \cup (0.5; 1)^2 \\ 1, & \text{otherwise;} \end{cases}$$

3. $K = K_3 = K_2 \psi(\mathcal{T}_h)$, $1 \leq \psi \leq 10^3$.



Smooth grids: Mimetic discretization

	1/h	Jacobi		Dendy		AMG		Kuznetsov	
		# itr	time	# itr	time	# itr	time	# itr	time
K_1	12	78	0.03	–	–	8	0.03	11	0.03
	24	167	0.26	14	0.18	10	0.12	13	0.12
	48	359	2.20	17	1.17	10	0.56	17	0.74
	96	749	23.7	20	6.32	12	2.91	20	4.36
	192	1547	215.	22	29.9	13	13.2	23	23.1
K_2	12	93	0.04	–	–	8	0.03	11	0.03
	24	195	0.30	18	0.22	11	0.12	15	0.14
	48	412	2.65	22	1.48	12	0.60	19	0.83
	96	862	27.4	28	8.68	12	2.90	22	4.76
	192	1769	247.	33	43.7	16	15.1	26	25.9
K_3	12	107	0.04	–	–	10	0.02	12	0.02
	24	243	0.37	27	0.33	11	0.11	16	0.14
	48	526	3.27	45	2.92	15	0.63	22	0.95
	96	1076	34.1	89	26.4	19	3.68	25	5.35
	192	2138	301.	175	223.	25	20.7	28	27.9

Smooth grids: Finite element method

		Jacobi		Dendy		AMG		Kuznetsov	
	$1/h$	# itr	time	# itr	time	# itr	time	# itr	time
K_1	12	94	0.03	—	—	11	0.02	14	0.02
	24	194	0.30	16	0.21	12	0.11	17	0.14
	48	403	2.50	20	1.34	17	0.69	19	0.83
	96	829	26.1	23	7.19	25	4.53	23	4.98
	192	1704	236.	26	35.0	37	27.2	25	25.2
K_2	12	113	0.04	—	—	11	0.02	14	0.02
	24	230	0.35	21	0.27	13	0.12	18	0.15
	48	471	2.93	26	1.73	17	0.69	20	0.86
	96	959	30.2	30	9.21	26	4.64	25	5.39
	192	1978	274.	37	48.9	42	30.3	27	27.0
K_3	12	123	0.05	—	—	8	0.02	13	0.02
	24	270	0.42	33	0.39	13	0.16	18	0.16
	48	560	3.51	58	3.72	18	0.70	21	0.91
	96	1117	35.5	111	32.7	23	4.14	25	5.37
	192	2255	313.	216	274.	30	23.1	28	28.0

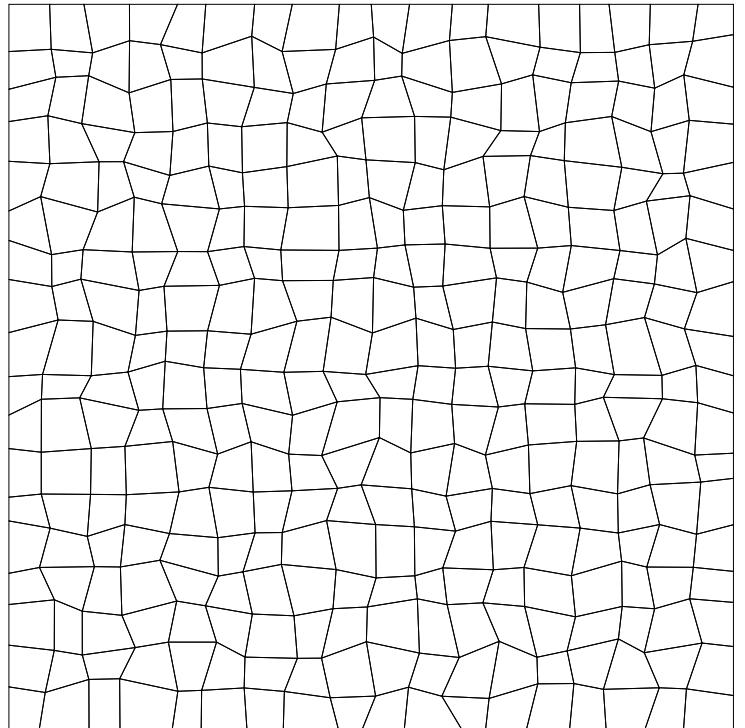
Random grids (1/3)

- random perturbation of mesh nodes preserving convexity of mesh cells;
- we consider both mimetic FD and mixed FE discretizations.

1. $K = K_1 = 1$;
2. $K = K_2(x)$ where

$$K_2 = \begin{cases} 10^3, & x \in (0; 0.5)^2 \cup (0.5; 1)^2 \\ 1, & \text{otherwise;} \end{cases}$$

3. $K = K_3 = K_2 \psi(\mathcal{T}_h)$, $1 \leq \psi \leq 10^3$.



Random grids: Mimetic discretization

		Jacobi		Dendy		AMG		Kuznetsov	
	$1/h$	# itr	time	# itr	time	# itr	time	# itr	time
K_1	12	60	0.02	—	—	8	0.02	6	0.02
	24	120	0.19	9	0.12	8	0.08	7	0.07
	48	250	1.54	13	0.92	9	0.40	7	0.33
	96	474	15.1	21	6.65	10	2.14	8	1.86
	192	939	130.	40	52.6	11	10.2	10	10.6
K_2	12	78	0.03	—	—	7	0.01	6	0.01
	24	159	0.25	12	0.16	9	0.08	7	0.07
	48	320	1.98	16	1.11	9	0.40	8	0.37
	96	647	20.8	26	8.10	10	2.14	9	2.07
	192	1328	185.	47	61.3	11	10.2	10	10.6
K_3	12	88	0.04	—	—	8	0.02	7	0.01
	24	195	0.30	21	0.26	9	0.08	11	0.10
	48	398	2.50	37	2.43	10	0.41	13	0.58
	96	817	26.0	73	21.8	11	2.22	14	3.09
	192	1657	230.	143	182.	11	10.2	15	15.4

Random grids: Finite element method

		Jacobi		Dendy		AMG		Kuznetsov	
	$1/h$	# itr	time	# itr	time	# itr	time	# itr	time
K_1	12	92	0.03	—	—	6	0.01	12	0.02
	24	182	0.27	13	0.17	8	0.09	13	0.11
	48	334	2.07	16	1.10	8	0.45	13	0.58
	96	638	20.1	19	6.05	8	2.15	14	3.10
	192	1212	168.	28	37.5	10	10.7	14	14.5
K_2	12	115	0.04	—	—	6	0.01	13	0.02
	24	233	0.36	18	0.22	6	0.08	14	0.12
	48	462	2.86	21	1.41	7	0.41	14	0.62
	96	923	29.2	25	7.78	9	2.30	15	3.29
	192	1842	255.	34	45.2	10	10.7	15	15.4
K_3	12	126	0.05	—	—	6	0.01	12	0.02
	24	277	0.42	33	0.39	8	0.09	17	0.15
	48	552	3.43	58	3.71	11	0.48	15	0.66
	96	1099	35.1	112	33.3	12	2.54	20	4.34
	192	2180	303.	222	281.	14	12.4	19	19.3

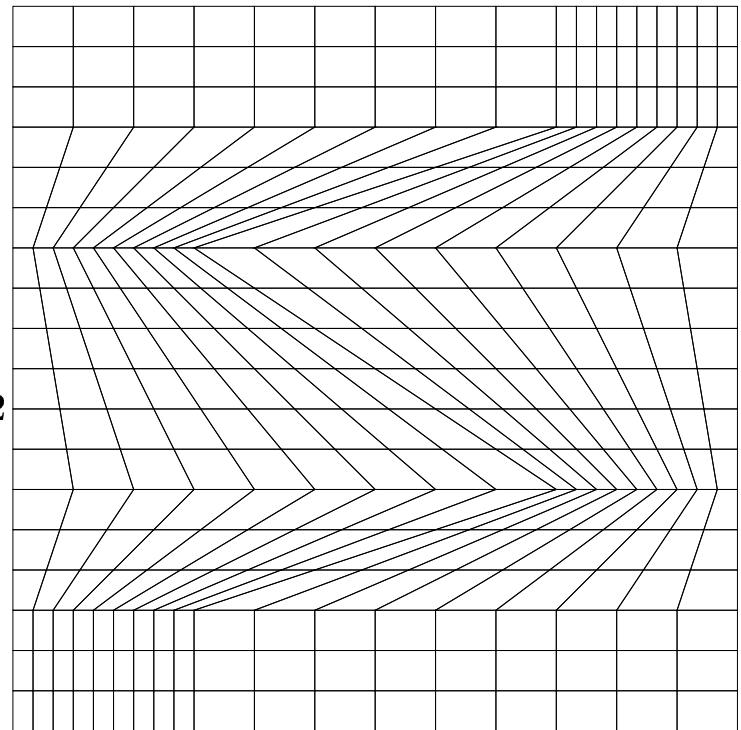
Kershaw grids (1/3)

- grids which are close to what we can expect in real applications;
- we consider both mimetic FD and mixed FE discretizations.

1. $K = K_1 = 1$;
2. $K = K_2(x)$ where

$$K_2 = \begin{cases} 10^3, & x \in (0; 0.5)^2 \cup (0.5; 1)^2 \\ 1, & \text{otherwise;} \end{cases}$$

3. $K = K_3 = K_2 \psi(\mathcal{T}_h)$, $1 \leq \psi \leq 10^3$.



Kershaw grids: Mimetic discretization

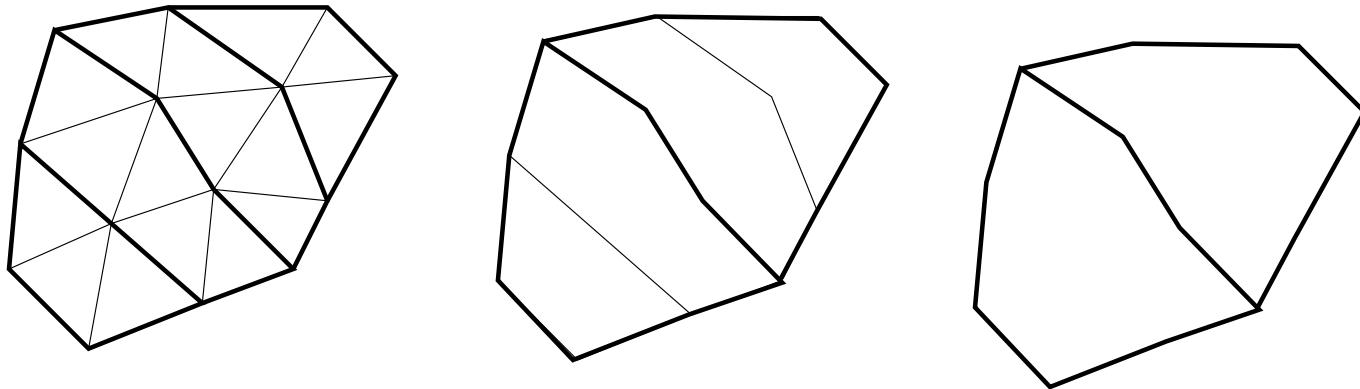
		Jacobi		Dendy		AMG		Kuznetsov	
	$1/h$	# itr	time	# itr	time	# itr	time	# itr	time
K_1	12	101	0.04	—	—	12	0.02	18	0.04
	24	212	0.32	24	0.29	16	0.15	26	0.22
	48	439	2.77	30	1.98	21	0.88	33	1.41
	96	911	29.1	37	11.3	24	4.76	41	8.69
	192	1877	261.	45	58.7	29	24.5	51	50.2
K_2	12	138	0.06	—	—	13	0.03	24	0.05
	24	276	0.43	37	0.44	20	0.17	33	0.28
	48	537	3.36	45	2.90	21	0.90	42	1.77
	96	1088	34.5	53	15.9	31	5.84	49	10.3
	192	2195	305.	61	78.8	28	23.6	58	57.0
K_3	12	160	0.06	—	—	18	0.03	25	0.04
	24	334	0.51	46	0.55	23	0.17	27	0.31
	48	676	4.26	63	4.02	31	1.09	50	2.12
	96	1364	43.3	110	32.5	35	6.09	55	11.6
	192	2698	375.	212	270.	39	30.5	66	64.5

Kershaw grids: Finite element method

		Jacobi		Dendy		AMG		Kuznetsov	
	$1/h$	# itr	time	# itr	time	# itr	time	# itr	time
K_1	12	103	0.05	—	—	16	0.03	18	0.03
	24	220	0.34	23	0.28	28	0.21	23	0.19
	48	463	2.87	30	1.97	49	1.69	28	1.20
	96	969	30.8	37	11.3	101	16.0	35	7.48
	192	2006	278.	45	58.7	209	140.	44	43.7
K_2	12	136	0.06	—	—	22	0.04	22	0.04
	24	277	0.16	36	0.43	34	0.25	29	0.25
	48	572	3.66	44	2.82	57	1.91	37	1.56
	96	1186	37.3	51	15.3	114	17.9	42	8.93
	192	2410	334.	62	80.2	232	155.	51	50.4
K_3	12	160	0.06	—	—	17	0.03	24	0.04
	24	343	0.52	44	0.52	29	0.22	32	0.27
	48	709	4.38	67	4.25	80	2.45	41	1.74
	96	1475	47.1	123	36.4	148	22.5	47	9.92
	192	2927	407.	245	311.	280	188.	56	54.9

Conclusion

1. In comparison with other preconditioners, the multilevel preconditioner was more robust on highly distorted meshes.
2. The method is easily generalized to unstructured grids:



3. The geometric nature of the preconditioner is very convenient for its parallel implementation.
4. A quality of the preconditioner depends on the tensor anisotropy.